Students must sign up to present one (or more) of these problems using the link distributed via email.

The problems below are each labeled "1", "2", or "3" in the margin. These roughly correspond to "Easy", "Medium", "Hard", and "Difficult". None of these are impossible for a student following along sufficiently in the course, but some problems will require more cleverness than others. Here is a rough guide to the difficulty markings.

1. This should be mostly mechanical, requiring little thought on the part of the student.
2. This may require deeper familiarity with the notes, or a simple but clever idea.
3. This requires solid understanding of topics in this course and maybe an additional skill (e.g. Calculus II).
4. This requires a deep understanding of topics in this course and may require skill in another lower-level mathematics course (e.g. Calculus II or Linear Algebra) as well.

[3] 1. Suppose $(A_n)_{n\in\mathbb{N}}$ is a sequence of real numbers satisfying the recursion $A_n = A_{n-1} + A_{n-2}$ for all $n \geq 2$. Prove $A_n = F_n A_1 + F_{n-1} A_0$ for all $n \geq 1$, where $F_n$ is the $n^{th}$ Fibonacci number.

[2] 2. Suppose $(a_n)_{n\in\mathbb{N}}$ is the sequence defined recursively by $a_0 = 0$, $a_1 = 1$, and $a_k = 2a_{k-1} - a_{k-2}$ for all $k \geq 2$. Conjecture and prove a formula for the $n^{th}$ term of the sequence.

[1] 3. How many positive divisors does the number $2^5 \cdot 3^2 \cdot 5^{10}$ have?

[1] 4. How many positive divisors does the number $2^9 \cdot 5^3 \cdot 7^2 \cdot 11^6$ have?

[2] 5. Suppose $n$ has prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. How many positive divisors does $n$ have?

[1] 6. How many elements of $[1,000,000]$ are divisible by either 7 or 11?

[1] 7. How many elements of $[1,000,000]$ are divisible by either 3 or 17 or 13?

[2] 8. How many elements of $[n]$ are divisible by either $p$ or $q$ for $p, q, n \in \mathbb{Z}_+$ with $\gcd(p,q) = 1$?

[1] 9. Prove that in any set of $n+1$ integers there are some two distinct elements with their difference divisible by $n$.

[2] 10. Let $S = (a_1, a_2, \cdots, a_n)$ be a sequence of integers with $n \geq 2$. Prove there are two runs of consecutive elements of $S$ with the same sum modulo $\dfrac{n^2+n-2}{2}$.

[2] 11. Prove that for every finite $\emptyset \neq S \subseteq \mathbb{Z}$ there is $\emptyset \neq T \subseteq S$ such that $\#S$ divides $\displaystyle\sum_{t\in T} t$.

[1] 12. Let $R = \{(x,y) : |x-y| < 1\} \subseteq \mathbb{Q} \times \mathbb{Q}$. Is $R$ reflexive? Symmetric? Transitive?

[1] 13. Prove $(a,b) \sim (c,d)$ when $a+d = b+c$ is an equivalence relation on $\mathbb{Z}_+ \times \mathbb{Z}_+$. What are its equivalence classes?

[1] 14. Prove $(a,b) \sim (c,d)$ when $ad = bc$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}_+$. What are its equivalence classes?

[3] 15. (Special Topic) Let $S$ denote the set of infinite sequences of rational numbers with the property that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n > N$ we have $|x_m - x_n| < \epsilon$. Prove the relation $(x_n) \sim (y_n)$ when $|x_n - y_n| \to 0$ as $n \to \infty$ is an equivalence relation on $S$. What are its equivalence classes?

[1] 16. Prove that the function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(n) = 2n$ is injective but not surjective.

[1] 17. Prove the function $f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by $f(x) = x^2$ is surjective but not injective.

[1] 18. Prove the function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) = x^2$ is bjective.

[1] 19. Construct a function $f\colon \mathbb{Z} \to \mathbb{Z}$ which is neither injective nor surjective.

[1] 20. Prove a function has an inverse if and only if it is bijective.

[1] 21. Prove if $g \circ f$ is injective, then $f$ is injective.

[1] 22. Prove if $g \circ f$ is surjective, then $g$ is surjective.

[1] 23. Suppose $g \circ f$ is bijective. Must $g$ or $f$ be bijective? Prove it or give a counterexample.

[1] 24. Let $f\colon A \to B$ be a function. Prove $x \sim y$ when $f(x) = f(y)$ is an equivalence relation on $A$.

[2] 25. (Special Topic) Let $\sim$ be an equivalence relation on set $S$. Prove that there is a set $T$ and a function $f\colon S \to T$ such that $x \sim y$ if and only if $f(x) = f(y)$.

[1] 26. Is there a graph $G$ with degree sequence $1 \leq 2 \leq 3 \leq 3 \leq 5$? Either construct $G$ or prove no such $G$ exists.

[1] 27. Is there a graph $G$ with degree sequence $1 \leq 1 \leq 1 \leq 2 \leq 3 \leq 5$? Either construct $G$ or prove no such $G$ exists.

[1] 28. Is there a graph $G$ with degree sequence $1 \leq 1 \leq 2 \leq 2 \leq 3 \leq 3$? Either construct $G$ or prove no such $G$ exists.

[2] 29. (Special Topic) Let $G$ be a graph. Prove the connection relation is an equivalence relation on $V(G)$.

[1] 30. Let $G$ be an $r$-regular graph on $n$ vertices. How many edges must $G$ have? Prove it.

[2] 31. (Special Topic) Let $0 < d_1 \leq d_2 \leq \cdots \leq d_m$ be a sequence of integers with sum $n$. Use graph theory to prove

$$\sum_{k=1}^{m} \binom{d_k}{2} \leq \binom{n}{2}.$$

[1] 32. For which $n$ is the graph $K_n$ Eulerian? For which $n$ is $K_n$ Hamiltonian? Prove it.

[1] 33. For which $m, n$ is the graph $K_{m,n}$ Eulerian? Prove it.

[1] 34. For which $m, n$ is the graph $K_{m,n}$ Hamiltonian? Prove it.

[1] 35. Is $1 \leq 1 \leq 1 \leq 2 \leq 2 \leq 2 \leq 3$ the degree sequence of a tree? Either prove it is not or exhibit such a tree.

[1] 36. Is $1 \leq 2 \leq 2 \leq 2 \leq 2 \leq 3$ the degree sequence of a tree? Either prove it is not or exhibit such a tree.

[2] 37. (Special Topic) Suppose $T$ is a tree with a vertex of degree $d > 0$. Prove $T$ has at least $d$ vertices of degree 1.

[2] 38. (Special Topic) Give an example of a graph $G$ with $\chi(G) = 4$ but $\omega(G) = 2$; prove your example is correct.

[1] 39. Prove every tree $T$ has $\chi(T) = 2$.

[1] 40. Compute (with proof!) the chromatic number of the Petersen graph.

[1] 41. If every proper induced subgraph of $G$ has chromatic number $d$, must $G$ have chromatic number $d$?

[1] 42. Prove that if $G$ is a graph with a perfect matching, then $G$ has an even number of vertices.

[1] 43. Produce a graph $G$ with an even number of vertices, satisfying the matching condition $\#N(S) \geq \#S$ for all $S \subseteq V(G)$, but which does not have a perfect matching.